Numerical stochastic perturbation theory for twisted reduced principal chiral model

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2021 2nd IITB-Hiroshima workshop Program (2021/10/25-27)

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Motivation & Introduction

The principal chiral model for $SU(N)$

Rank of matrix

$$Z = \int \prod_{\mu,n} dU_\mu(n) \exp\left\{ -bN_c \operatorname{Tr}\left(\delta_\mu U(n) \delta_\mu U^\dagger(n) \right) \right\}$$

with

$$\delta U(n) = U(n+\mu) - U(n) \quad \beta = bN_c$$

- Two dimensional $\mathrm{SU}(N_c) \times \mathrm{SU}(N_c)$ principal chiral model - asymptotically free
- Non-perturbative studies on the lattice
  - Finite volume correction $\langle \mathcal{O}(L) \rangle = \langle \mathcal{O}^\infty \rangle - \frac{F(L)}{L^2} + \mathcal{O}(L^4)$
- Perturbative calculation on the lattice when $N \to \infty$ limit hold
- QCD like theory
- Too expensive L^2 $SU(N)$ matrix for one configuration

Motivation & Introduction

Twisted Eguchi-Kawai reduction : from lattice to single site

$$S = -bN_c \sum_{\mu} \mathrm{Tr}\left(V_\mu V_\mu^\dagger\right)$$

with

$$V_\mu \equiv \Gamma_\mu U \Gamma_\mu^\dagger - U$$

$$\Gamma_1 \Gamma_2 = \exp\left\{\frac{2\pi i K}{N_c}\right\} \Gamma_2 \Gamma_1 \quad \Gamma_\mu \in SU(N_c)$$

- Volume reduction was held in the large-N limit
- Tuning the twist parameter K
  - K, N co-prime
  - $\frac{K}{N} > \Lambda_{cut}$
- Finite-N corrections ↔ finite-size effects ($L = N_c$)

Motivation & Introduction

- A method that may be able to deal with the perturbation calculation on the lattice up to high-order terms is Numerical Stochastic Perturbation Theory (NSPT)
  - Large N
  - High-order coefficient
- In QFT the asymptotic nature of an expansion in powers of the coupling constant α suggests a factorial divergence occurs when we include high-order coefficients.

Numerical stochastic perturbation theory

- Monte Carlo simulation: Evaluation of Euclidean path integrals for field theories.
  - Partition function $Z = \int D\phi \exp[-S[\phi]]$ $\langle 0 \rangle = \frac{1}{Z} \int D\phi O[\phi] \exp[-S[\phi]]$

Expectation value of observable = Evaluate multi-dimensional integral

  - Probabilistic density function $P[\phi] = \frac{1}{Z} \exp[-S[\phi]]$ $\langle 0 \rangle = \int D\phi O[\phi] P[\phi]$

- Numerical application of stochastic quantization (G.Parisi,Y-S.Wu,1981)

π : Generated from Gaussian noise

  - Langevin equation => molecular dynamics based
  - Adding an extra DoF $\phi(x) \rightarrow \phi(x,t)$
  - Infinite stochastic time limit

$$H = \frac{\pi^2}{2} + S[\phi] \qquad P'[\pi, \phi] = \frac{1}{Z'} \exp\left[-\frac{\pi^2}{2} - S[\phi]\right]$$

$$\dot{\pi} = -\frac{\partial S}{\partial \phi} = F, \dot{\phi} = \pi \qquad \langle O \rangle = \int D\pi D\phi O[\phi] P'[\phi]$$

$$\left\langle O\left[\phi_\eta\left(x_1;t\right)\ldots\phi_\eta\left(x_n;t\right)\right]\right\rangle_\eta \rightarrow_{t\to\infty} \left\langle O\left[\phi\left(x_1\right)\ldots\phi\left(x_n\right)\right]\right\rangle$$

Numerical stochastic perturbation theory

- Evaluate the perturbation expansion of physical quantities

$$S[\phi] = S_0[\phi] + S_{Int}[\phi, g] \qquad \phi = \sum_{k=0}^{N_{trun}} g^k \phi^{(k)}, \pi = \sum_{k=0}^{N_{trun}} g^k \pi^{(k)}$$

- Hierarchical system of partial differential equations

$$\dot{\pi}^{(k)} = -\frac{\partial S^{(k)}}{\partial \phi}\left[\phi^{(0)}, \phi^{(1)}, \ldots, \phi^{(k-1)}\right], \quad \dot{\phi}^{(k)} = \pi^{(k)} \quad k = 0,1,2,\ldots$$

$$\langle 0 \rangle = \sum_{k=0} g^k \left\langle O^{(k)} \right\rangle = \lim_{T\to\infty} \frac{1}{T} \sum_{k=0} g^k O^{(k)} \left[\phi^{(0)}(t), \phi^{(1)}(t), \ldots, \phi^{(k)}(t)\right]$$

NSPT for matrix model

- Convolution operator $\circledast$

$$A = \sum_{k=0} g^k A^{(k)} = A^{(0)} + gA^{(1)} + g^2 A^{(2)} + \cdots \qquad B = \sum_{k=0} g^k B^{(k)} = B^{(0)} + gB^{(1)} + g^2 B^{(2)} + \cdots$$

$$AB = C = \sum_{k=0}^{N_{trun}} g^k C^{(k)} = C^{(0)} + gC^{(1)} + g^2 C^{(2)} + \cdots \qquad C^{(l)} = (A \circledast B)^{(l)} = \sum_{k=0}^{l} A^{(l-k)} B^{(k)}$$

- Expansion for HMD equation(matrix model)

$$\dot{U}^{(i)} = i(\pi \circledast U)^{(i)}$$
$$\dot{\pi}^{(i)} = F^{(i)}$$

- Expansion for $F^{(i)}$

$$F^{(k)} = iN_c \left(V^{(k)} - \frac{1}{N_c} \mathrm{Tr}[V^{(k)}] \right)$$
$$V^{(k)} = S^{(k)} - S^{(k)\dagger}$$
$$S^{(k)} = 4U^{(k)} + X^{(k)} + (U \circledast X)^{(k)} \qquad k = 1 \ldots N_{truncated}$$
$$X^{(k)} = \sum_{\mu=1}^{2} \left[\Gamma_\mu^\dagger U^{(k)\dagger} \Gamma_\mu + \Gamma_\mu U^{(k)\dagger} \Gamma_\mu^\dagger \right] \qquad k = 0..N_{truncated}$$

NSPT for matrix model

- Workflow
  - Test NSPT for few $N_c$and a proper K
  - Calculate internal energy up to $\mathcal{O}(g^8)$
  - Perform $1/N_c^2$ fitting

$$E = \frac{1}{2N_c} \sum_{\mu=1,2} \mathrm{Re}\,\mathrm{Tr}\left[U\Gamma_u U^\dagger \Gamma_u^\dagger\right]$$

- Few points that need attention
  - The problem with ergodicity in lead order
    - Randomization the trajectory length
    - Partial momentum refreshing scheme $\pi_\mu = c_1\pi_\mu + \mu + \sqrt{1-c_1^2}\eta_\mu$
  - Computationally intensive part : convolution operation
    - Naive algorithm $\propto \mathcal{O}(N_{trun}^2)$
    - FFT based convolution $\propto \mathcal{O}(N_{trun} log N_{trun})$
  - Violation of special unitarity
    - Check the norm of targeted matrix
    - Implement re-unitalization scheme

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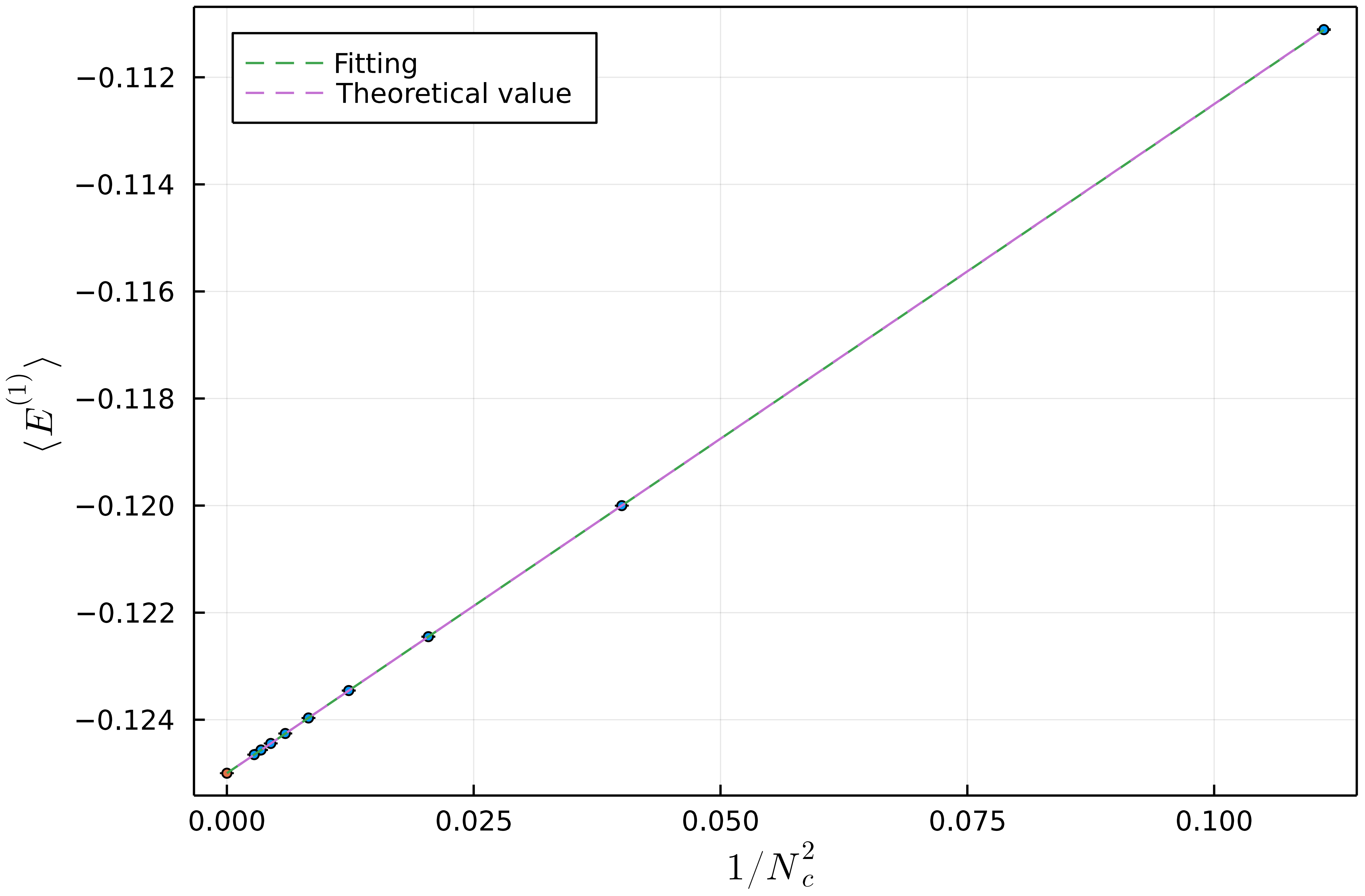
Numerical results

1 loop $\mathcal{O}(b)$ case (no volume dependency)

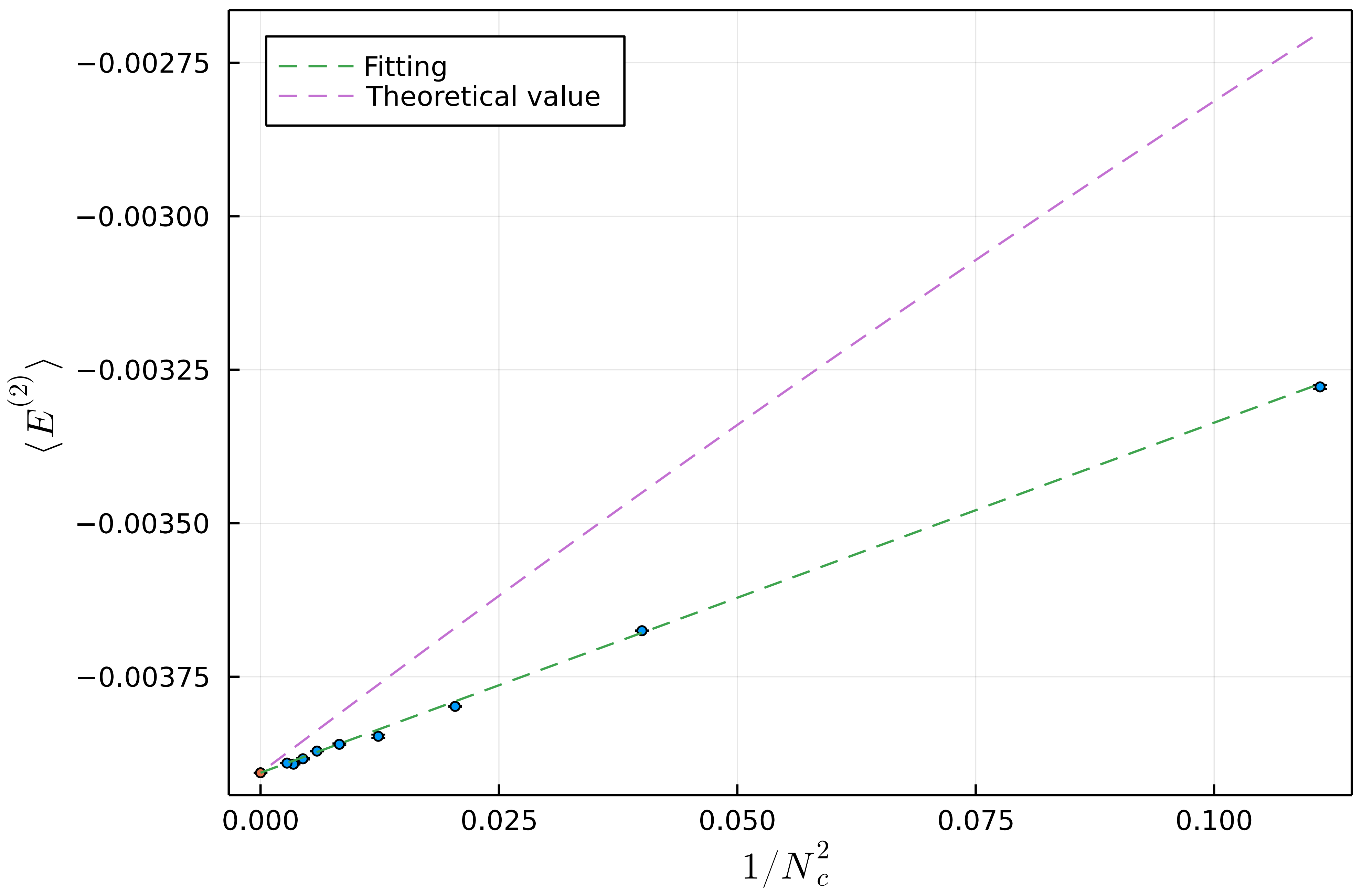


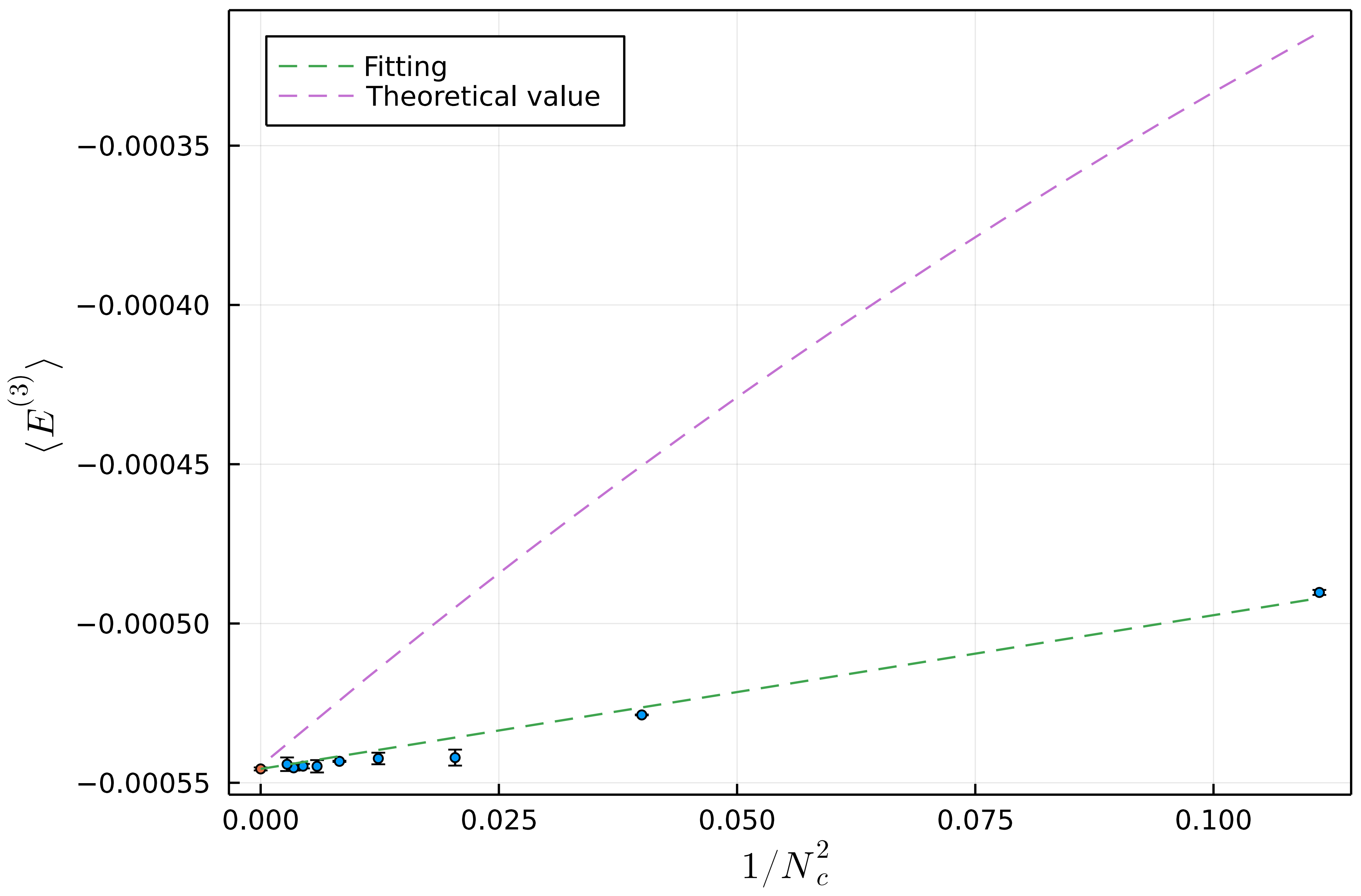
	Theoretical value	NSPT
$E^{(1)}$	-1/8	-0.124999(1)

Numerical results

2 and 3 $\mathcal{O}(b^{2,3})$ loop case (volume dependency)

	Theoretical value	NSPT
$E^{(2)}$	-0.00390625	-0.00390637(23)
$E^{(3)}$	-0.000544	-0.000545(4)

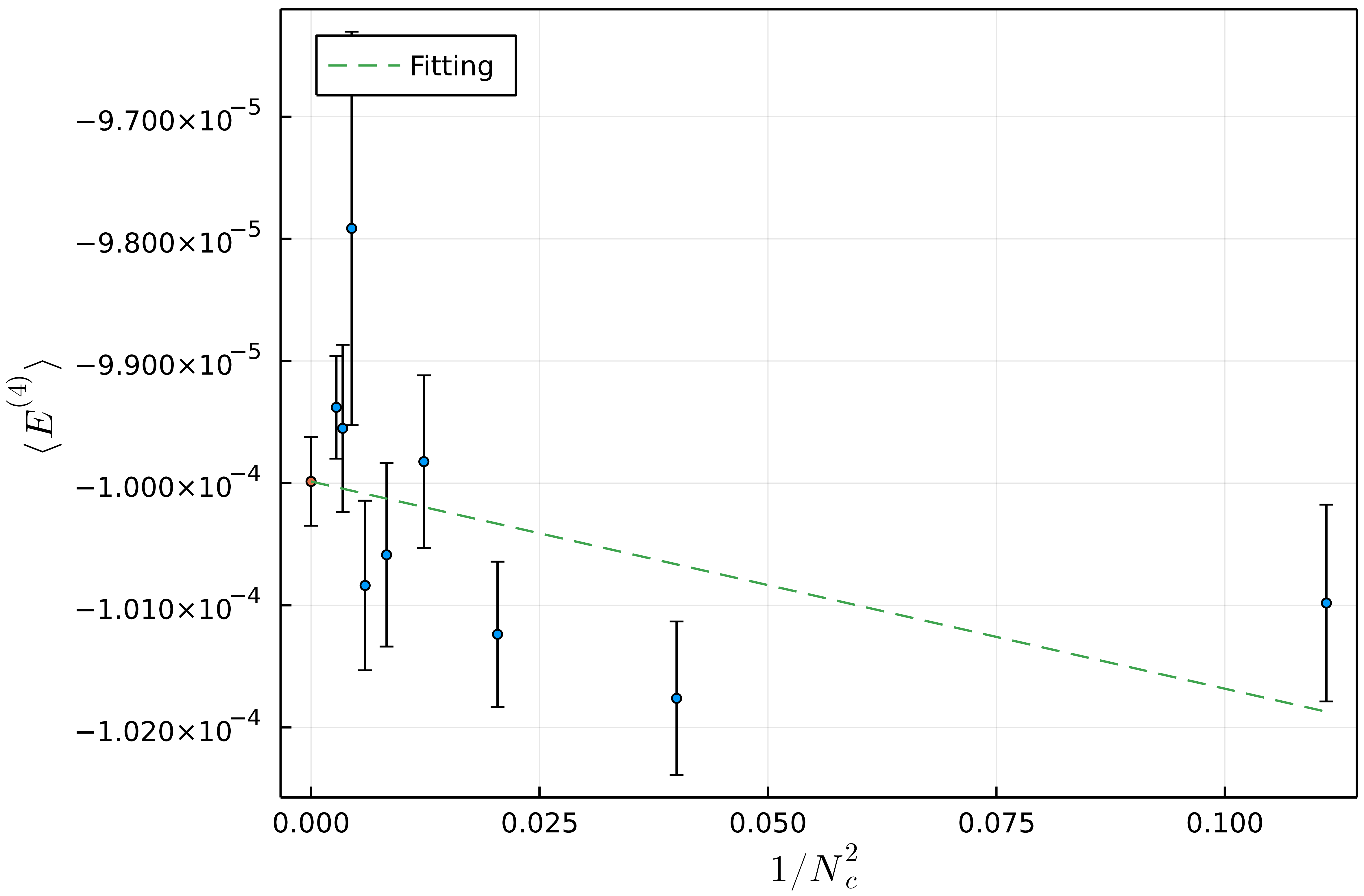




Numerical results

4 loop $\mathcal{O}(b^4)$ result

- Our result for four loop case is $E^{(4)} = -0.00009998(36)$

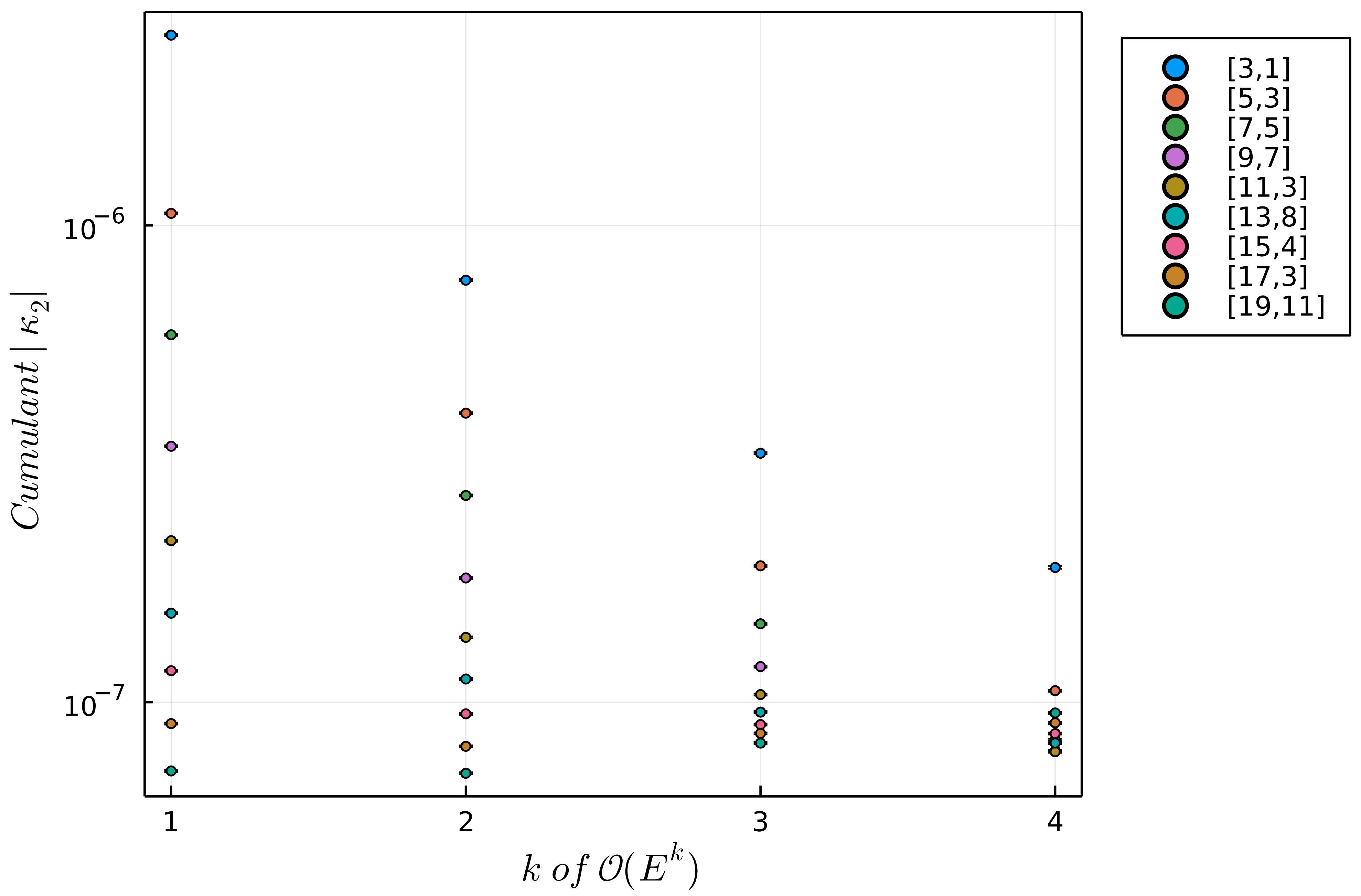


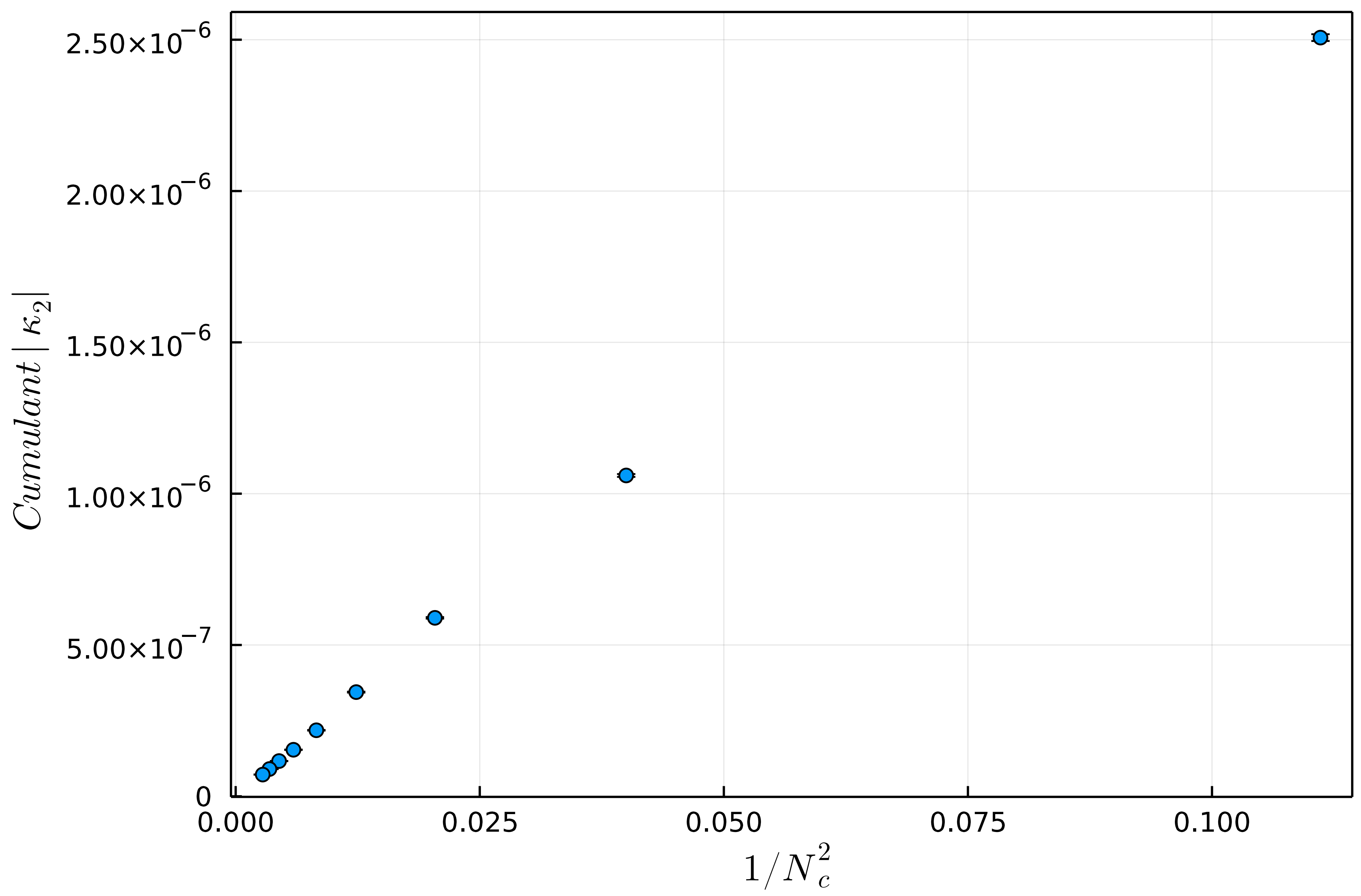
Numerical results

Factorization

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle + \mathbf{O}(1/N_c^2)$$

$$\kappa_2 = var(E) = \langle E^2 \rangle - \langle E \rangle^2 \Rightarrow 0 \quad \text{as} \quad N_c \to \infty$$





Summary

- Calculation up to $\mathcal{O}(g^8)$ show the feasibility of combining NSPT and TRPCM.
- The value of the first three coefficients matches very precisely with its theoretical values in the large-N limit.
- The 2 and 3 loop result shows how the volume dependency was eliminated as $N_c \to \infty$
- 4 loop coefficient with considerable precision.

END

THANK YOU FOR YOUR

WATCHING